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Computability by monadic second-order logic

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ABSTRACT

A binary relation on graphs is recursively enumerable if and only if it can be computed by a formula of monadic second-order logic. The latter means that the formula defines a set of graphs, in the usual way, such that each “computation graph” in that set determines a pair consisting of an input graph and an output graph.

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There are many characterizations of computability, but the one presented here does not seem to appear explicitly in the literature.¹ Nevertheless, it is a natural and simple characterization, based on the intuitive idea that a computation of a machine, or a derivation of a grammar, can be represented by a graph satisfying a formula of monadic second-order (MSO) logic. Assuming the reader to be familiar with MSO logic on graphs (see, e.g., [3, Chapter 5]), the MSO-computability of a binary relation on graphs can be given in half a page, see below. One advantage of the definition is that there is no need to code the graphs as strings or numbers.

For an alphabet Ψ , we consider directed edge-labeled graphs $g = (V, E)$ over Ψ where V is a nonempty finite set of nodes and $E \subseteq V \times \Psi \times V$ is a set of labeled edges. We also denote V by V_g , and E by E_g . An edge $(u, \psi, v) \in E_g$ is called a ψ -edge. Isomorphic graphs are considered to be equal. The set of all (abstract) graphs over Ψ is denoted by \mathcal{G}_Ψ .

To model computations we use a special edge label ν that is not in Ψ . We define a *computation graph* over Ψ to

be a graph h over $\Psi \cup \{\nu\}$ with at least one ν -edge such that for every $u, v, u', v' \in V_h$,

- (1) $(u, \nu, u) \notin E_h$, and
- (2) if $(u, \nu, v), (u', \nu, v') \in E_h$, then $(u, \nu, v') \in E_h$.

The *input graph* $\text{in}(h)$ is defined to be the subgraph of h induced by all nodes that have an outgoing ν -edge, and the *output graph* $\text{out}(h)$ is the subgraph of h induced by all nodes that have an incoming ν -edge. By (2) above, the ν -edges of h connect every node of $\text{in}(h)$ to every node of $\text{out}(h)$, and so by (1) above, $V_{\text{in}(h)}$ and $V_{\text{out}(h)}$ are disjoint. In fact, the role of the ν -edges is just to specify an ordered pair of disjoint subsets of V_h , in a simple way. Note that there may be arbitrarily many nodes and edges in h that belong neither to $\text{in}(h)$ nor to $\text{out}(h)$. Also, there may be edges between $\text{in}(h)$ and $\text{out}(h)$ other than the ν -edges. This notion of computation graph generalizes the “pair graph” of [9], which on its turn generalizes the “origin graph” of [1].

For a set H of computation graphs over Ψ we define the *graph relation computed by H* to be $\text{rel}(H) = \{(\text{in}(h), \text{out}(h)) \mid h \in H\} \subseteq \mathcal{G}_\Psi \times \mathcal{G}_\Psi$. Finally, for an alphabet Γ , we say that a graph relation $R \subseteq \mathcal{G}_\Gamma \times \mathcal{G}_\Gamma$ is *MSO-computable* if there are an alphabet Δ and an MSO-definable set H of computation graphs over $\Gamma \cup \Delta$ such

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¹ This first sentence and the first part of the next sentence are taken over from [8].

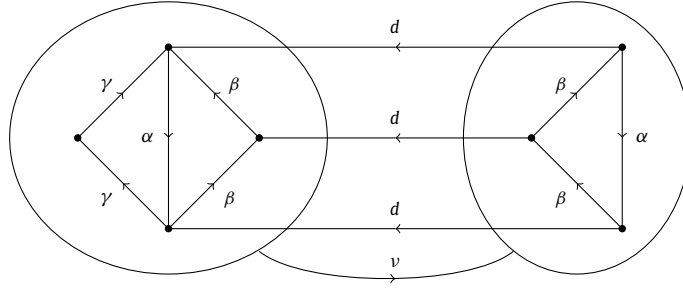


Fig. 1. A computation graph h for an induced subgraph, with $\Gamma = \{\alpha, \beta, \gamma\}$. The input graph $\text{in}(h)$ and output graph $\text{out}(h)$ are surrounded by ovals, and the v -labeled edge from the left oval to the right oval represents the 12 v -labeled edges from each node of $\text{in}(h)$ to each node of $\text{out}(h)$.

that $\text{rel}(H) = R$. As observed before, we assume the reader to be familiar with MSO logic on graphs.² The closed MSO formula φ that defines the set H can be viewed as a “machine” of which the computations are represented by the graphs in H . We will also say that $\text{rel}(H)$ is the graph relation computed by φ . For each $h \in H$, the input graph $\text{in}(h)$ and the output graph $\text{out}(h)$ must be graphs over the input/output alphabet Γ . The *auxiliary alphabet* Δ is needed to allow the edges of a computation graph that are not part of its input or output graph, to carry arbitrary information in their label; it is similar to the “working alphabet” of a machine. This notion of MSO-computability generalizes the “MSO-expressibility” of graph relations of [9],³ which on its turn generalizes the MSO graph transductions of [3, Chapter 7] (as shown in [9, Section 7.1]).

Examples. (1) Let $R \subseteq \mathcal{G}_\Gamma \times \mathcal{G}_\Gamma$ be the set of all (g, g') such that g' is an induced subgraph of g . The graph relation R is MSO-computable because it can be computed by an MSO-definable set H of computation graphs over $\Gamma \cup \Delta$, with $\Delta = \{d\}$. We note that, by definition, the set of all computation graphs h over $\Gamma \cup \Delta$ is MSO-definable, and the sets of nodes $V_{\text{in}(h)}$ and $V_{\text{out}(h)}$ can be expressed in MSO logic. The set H consists of computation graphs h such that $V_h = V_{\text{in}(h)} \cup V_{\text{out}(h)}$, $\text{in}(h)$ and $\text{out}(h)$ are graphs over Γ , and the d -edges form an isomorphism from $\text{out}(h)$ to an induced subgraph of $\text{in}(h)$. The last condition means, in detail, that for every $u, v, u', v' \in V_h$,

- if (u, d, v) is an edge of h , then $u \in V_{\text{out}(h)}$ and $v \in V_{\text{in}(h)}$,
- if $u \in V_{\text{out}(h)}$, then u has an outgoing d -edge,
- if (u, d, v) and (u', d, v') are edges of h , then
 - $u = u'$ if and only if $v = v'$, and
 - for every $\gamma \in \Gamma$, $(u, \gamma, u') \in E_h$ if and only if $(v, \gamma, v') \in E_h$.

There may be γ -edges in h between $\text{in}(h)$ and $\text{out}(h)$, with $\gamma \in \Gamma$; though they are harmless, we could additionally forbid them. For an example of such a computation

graph see Fig. 1. Obviously the above conditions can be expressed by an MSO formula φ , which defines H . Moreover $\text{rel}(H) = R$, and hence R is MSO-computable. Note that R is even “MSO-expressible”, in the sense of [9].

As another (similar) example, if R consists of all (g, g') such that g has at least two, disjoint, induced subgraphs isomorphic to g' , then we take $\Delta = \{d_1, d_2\}$, we require that the d_i -edges satisfy the same conditions as the d -edges above (for each $i \in \{1, 2\}$), and we require that no node of $\text{in}(h)$ has both an incoming d_1 -edge and an incoming d_2 -edge.

(2) Let g_0 be a fixed graph over Γ , and let $R \subseteq \mathcal{G}_\Gamma \times \mathcal{G}_\Gamma$ be the set of all (g, g_0) such that the number of nodes of g with an outgoing α -edge equals its number of nodes with an outgoing β -edge, with $\alpha, \beta \in \Gamma$. There is an MSO-definable set H of computation graphs over $\Gamma \cup \Delta$ such that $\text{rel}(H) = R$, where $\Delta = \{d, e\}$. It consists of all graphs h that are obtained by adding v -, d - and e -edges to the disjoint union of g, g' , and g_0 , where g is an arbitrary graph over Γ and g' is isomorphic to g . The v -edges determine that $\text{in}(h) = g$ and $\text{out}(h) = g_0$. The d -edges establish an isomorphism between g and g' , and the e -edges establish a bijection between the nodes of g with an outgoing α -edge and the nodes of g' with an outgoing β -edge. Since these requirements can easily be expressed in MSO logic, R is MSO-computable. It is not difficult to show that R is not “MSO-expressible”, cf. the Conclusion of [9]. \square

Our aim is now to prove the following theorem.

Theorem. A graph relation is MSO-computable if and only if it is recursively enumerable.

Recursive enumerability of a graph relation R means that there is a (single tape) nondeterministic Turing machine M such that $(g, g') \in R$ if and only if, on input g , M has a computation that outputs g' . In one direction this theorem is obvious: every MSO-computable graph relation is recursively enumerable. In fact, on input $g \in \mathcal{G}_\Gamma$ (coded as a string in an appropriate way) M guesses a computation graph h over $\Gamma \cup \Delta$ such that $\text{in}(h) = g$, checks whether h satisfies the MSO formula φ (cf. [3, Chapter 6]), and if so, outputs the (coded) graph $\text{out}(h)$. To show the other direction we first consider the case of string relations. For the notion of MSO-computability we represent a string $w = \gamma_1 \gamma_2 \dots \gamma_k$ over Γ by the graph $\text{gr}(g) \in \mathcal{G}_\Gamma$ such that $V_{\text{gr}(g)} = \{1, 2, \dots, k+1\}$ and $E_{\text{gr}(g)} = \{(j, \gamma_j, j+1) \mid$

² The atomic formulas of MSO logic are $x = y$, $x \in X$, and $\text{edge}_\psi(x, y)$, where x and y are nodes, X is a set of nodes, and $\text{edge}_\psi(x, y)$ expresses that there is a ψ -edge from x to y .

³ The relation R is “MSO-expressible”, in the sense of [9, Section 3.1], if it is MSO-computable by a set H of pair graphs, where a pair graph is a computation graph h such that $V_h = V_{\text{in}(h)} \cup V_{\text{out}(h)}$.

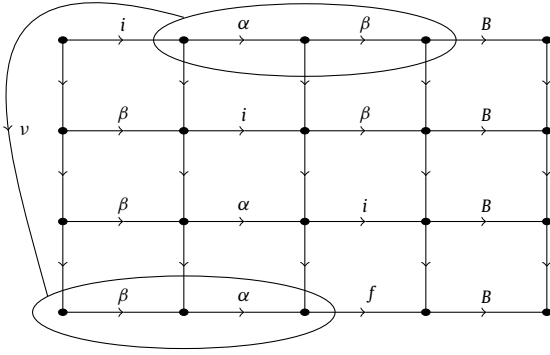


Fig. 2. A computation graph h over $\Gamma \cup \Delta$, with $\Gamma = \{\alpha, \beta\}$ and $\Delta = \{i, f, B, *\}$. It represents the computation of a (very simple) Turing machine M that changes every α of the input string into β and vice versa. Here the input string is $\alpha\beta$, and M uses space $m = 3$ and time $n = 4$. The initial state of M is i , the final state is f , and the instructions are $i\alpha \vdash \beta i$, $i\beta \vdash \alpha i$, and $iB \vdash fB$, where B is the blank. The strings w_1, w_2, w_3, w_4 corresponding to M 's computation are $i\alpha\beta B, \beta i\beta B, \beta \alpha i B$, and $\beta \alpha f B$. The $*$ -labels of the vertical edges of h are omitted.

$1 \leq j \leq k$). The proof is similar to the one of [3, Theorem 5.6]. Let M be a nondeterministic Turing machine that computes the recursively enumerable string relation $R \subseteq \Gamma^* \times \Gamma^*$. Consider a computation of M that, for an input string w , outputs the string w' . Suppose that it uses space m and time n . Thus, it can be viewed as a sequence of strings w_1, \dots, w_n , each of length $m + 1$, such that w_i is the content of M 's tape at time i (including the state of M), w_1 contains w (plus the initial state and blanks), and w_n contains w' (and a final state and blanks). Clearly, this sequence can be represented by a grid of dimension $n \times (m + 2)$. The rows of the grid are the graphs $\text{gr}(w_1), \dots, \text{gr}(w_n)$, which are connected by $*$ -labeled column edges from the j -th node of w_i to the j -th node of w_{i+1} for every $1 \leq i \leq n - 1$ and $1 \leq j \leq m + 2$. It is easy to turn that grid into a computation graph h by adding v -edges from the nodes of $\text{gr}(w)$ in the first row to those of $\text{gr}(w')$ in the last row. Thus, h is a computation graph over $\Gamma \cup \Delta$ such that $\text{in}(h) = \text{gr}(w)$ and $\text{out}(h) = \text{gr}(w')$, where the alphabet Δ consists of the column symbol $*$, the working symbols of M (including the blank), and the states of M . For an example see Fig. 2. Since the set of grids is MSO-definable (as shown in [3, Section 5.2]), it is a straightforward exercise in MSO logic to show that the computation graphs h , obtained from the (successful) computations of M , can be defined by an MSO formula φ_M . In particular, φ_M should express that the consecutive rows of the grid (corresponding to strings w_i and w_{i+1}) satisfy the (local) changes determined by the instructions of M . This shows that the graph relation computed by φ_M is $\text{gr}(R) = \{(\text{gr}(w), \text{gr}(w')) \mid (w, w') \in R\}$, and so, $\text{gr}(R)$ is MSO-computable.

For an alphabet Γ , let the *graph encoding relation* enc_Γ consist of all pairs $(g, \text{gr}(w))$ such that $g \in \mathcal{G}_\Gamma$ and w is an appropriate encoding of g as a string (which we will specify later).⁴ By definition, if a graph relation $R \subseteq \mathcal{G}_\Gamma \times \mathcal{G}_\Gamma$

is recursively enumerable then there is a recursively enumerable string relation R' such that R is the composition of enc_Γ , $\text{gr}(R')$, and enc_Γ^{-1} . Hence, to obtain our theorem for graph relations it now suffices to prove the following two lemmas.

Lemma 1. *The class of MSO-computable graph relations is closed under inverse and composition.*

Lemma 2. *For every Γ , the graph encoding relation enc_Γ is MSO-computable.*

Proof of Lemma 1. Closure under inverse is obvious: just reverse the direction of all v -edges. To prove closure under composition, let R_1 and R_2 be graph relations computed by MSO formulas φ_1 and φ_2 . We may assume that φ_1 and φ_2 use the same auxiliary alphabet Δ . Moreover, we may assume that every computation graph h defined by φ_1 or φ_2 is connected: if not, then add a special symbol μ to Δ and require that every node u of h that is not in $\text{in}(h)$ or $\text{out}(h)$, has a μ -edge to $\text{in}(h)$ or $\text{out}(h)$. Finally, we assume that φ_1 uses the label v_1 instead of v , and φ_2 uses v_2 instead of v , with $v_1 \neq v_2$. The MSO formula φ that computes the composition of R_1 and R_2 , uses the auxiliary alphabet $\Delta \cup \{v_1, v_2, d\}$ and defines computation graphs h that are obtained as the disjoint union of a computation graph h_1 of φ_1 and a computation graph h_2 of φ_2 , enriched by d -edges that establish an isomorphism between $\text{out}(h_1)$ and $\text{in}(h_2)$, and by v -edges from $\text{in}(h_1)$ to $\text{out}(h_2)$. It should be clear that this can be realized by φ ; for instance, it expresses that the connected components of h minus its enriching edges satisfy φ_1 or φ_2 , depending on whether they contain a v_1 -edge or a v_2 -edge. \square

Proof of Lemma 2. We first specify the relation enc_Γ . Let $g \in \mathcal{G}_\Gamma$. We may assume that V_g is the set of strings $\{a, a^2, \dots, a^n\}$ over the alphabet $\{a\}$, for some $n \geq 1$, where $a \notin \Gamma$. Let $E_g = \{(u_1, \gamma_1, v_1), \dots, (u_m, \gamma_m, v_m)\}$ for some $m \geq 0$. We encode g , in a standard way, as the string

$$w = \#a\#a^2\#\dots\#a^n\$u_1\gamma_1v_1\$ \dots \$u_m\gamma_mv_m\$$$

over the alphabet $\Omega = \Gamma \cup \{a, \#, \$\}$, and we define the graph encoding relation $\text{enc}_\Gamma \subseteq \mathcal{G}_\Gamma \times \mathcal{G}_\Omega$ to consist of all pairs $(g, \text{gr}(w))$. Note that since w depends on linear orderings of V_g and E_g , a graph g has in general more than one encoding. On the other hand, the relation enc_Γ^{-1} is a function. The set of strings over Ω that encode graphs over Γ is not a regular language, and hence the set $\text{enc}_\Gamma(\mathcal{G}_\Gamma)$ of graphs over Ω is not MSO-definable [2,6,12]. However, by enriching each $\text{gr}(w)$ with α -edges and δ -edges (where α and δ are special symbols not in Ω), we can turn $\text{enc}_\Gamma(\mathcal{G}_\Gamma)$ into an MSO-definable set of graphs. For a string w as displayed above we define $\text{gr}^+(w)$ to be the graph $\text{gr}(w)$ to which α -edges and δ -edges are added as follows. For an example see Fig. 3. The α -edges allow an MSO formula to express the fact that

⁴ Appropriateness means that the encoding and the corresponding decoding are computable in a straightforward intuitive sense. In particular, it

is decidable whether or not a given string is the encoding of some graph. Any standard encoding of graphs satisfies these requirements.

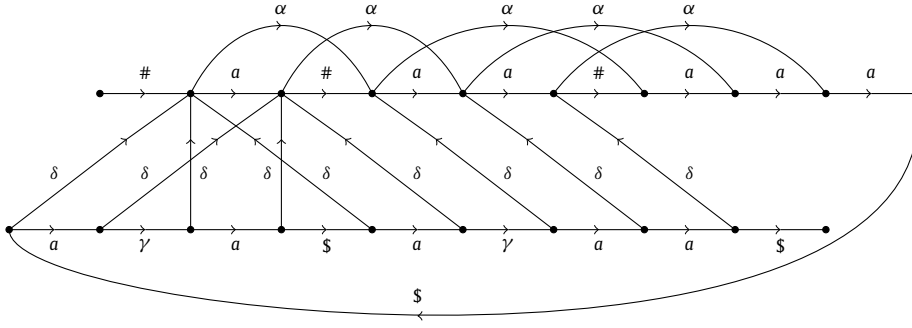


Fig. 3. The graph $\text{gr}^+(w)$ for the string $w = \#a\#aa\#aaa\$a\gamma a\$a\gamma a\$a$, which is an encoding of the graph g with $V_g = \{a, aa, aaa\}$ and $E_g = \{(a, \gamma, a), (a, \gamma, aa)\}$. The graph $\text{gr}(w)$ is obtained from $\text{gr}^+(w)$ by removing all α - and δ -edges.

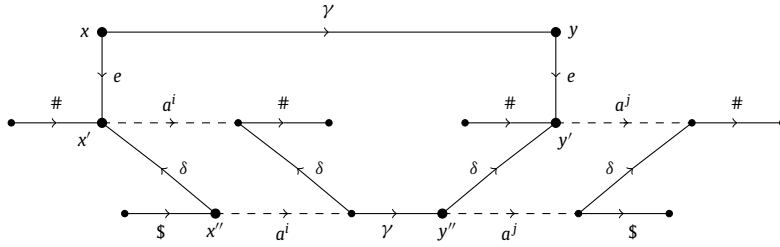


Fig. 4. Parts of a computation graph h showing the MSO-computability of enc_Γ . The nodes x and y belong to $\text{in}(h)$, all other nodes to $\text{mid}(h)$.

the first half of w is of the form $\#a\#a^2\#\dots\#a^n\$$. For each substring $\#a^i\#a^j$ of w ($1 \leq i \leq n-1$) there are α -edges in $\text{gr}^+(w)$ from the nodes of the first occurrence of $\text{gr}(a^i)$ in $\text{gr}(w)$ to the nodes of the second occurrence of $\text{gr}(a^i)$ in $\text{gr}(w)$, such that they form an isomorphism between these two subgraphs. An MSO formula on $\text{gr}^+(w)$ can express that w is in the regular language $\#a(\#a^*)^*(\$a^*\Gamma a^*)^*\$,$ and, using the outgoing α -edges of $\text{gr}(\#a^i\#)$, it can enforce that each substring $\#a^i\#$ is followed by $a^{i+1}\#$ or $a^{i+1}\$$. The δ -edges in $\text{gr}^+(w)$ witness the fact that for each substring $\$u_j\gamma_jv_j\$$ of w ($1 \leq j \leq m$) both u_j and v_j are in $\{a, a^2, \dots, a^n\}$, i.e., u_j and v_j are “declared” in the first half of w . Thus, there are δ -edges from the nodes of $\text{gr}(u_j)$ to the nodes of some $\text{gr}(\#a^i\#)$ or $\text{gr}(\#a^i\$)$ in the first half of $\text{gr}(w)$ that establish an isomorphism between $\text{gr}(u_j)$ and $\text{gr}(a^i)$, and similarly for $\text{gr}(v_j)$. This can also easily be expressed by an MSO formula. Moreover, the δ -edges can be used to express that an edge is not encoded twice in w , i.e., if $j \neq k$ then $\$u_j\gamma_jv_j\$ \neq \$u_k\gamma_kv_k\$$; in fact, $u_j = u_k$ if and only if the two δ -edges that start from the first nodes of $\text{gr}(u_j)$ and $\text{gr}(u_k)$ in $\text{gr}^+(w)$, lead to the same node (and similarly for $v_j = v_k$). We now define enc_Γ^+ to consist of all pairs $(g, \text{gr}^+(w))$ where w encodes g . It follows that the set $\text{enc}_\Gamma^+(\mathcal{G}_\Gamma)$ is MSO-definable.⁵

Finally, we show that $\text{enc}_\Gamma \subseteq \mathcal{G}_\Omega \times \mathcal{G}_\Omega$ is MSO-computable by describing the computation graphs h over $\Omega \cup \Delta$ in an MSO-definable set H such that $\text{rel}(H) = \text{enc}_\Gamma$. The auxiliary alphabet is $\Delta = \{\alpha, \delta, d, e\}$. Let $\text{mid}(h)$ be the subgraph of h induced by the nodes of h that are not

incident with a v -edge, i.e., that are not in $V_{\text{in}(h)}$ or $V_{\text{out}(h)}$. First, we require that $\text{mid}(h)$ is in $\text{enc}_\Gamma^+(\mathcal{G}_\Gamma)$, i.e., $\text{mid}(h) = \text{gr}^+(w)$ where w encodes some graph g in \mathcal{G}_Γ . Second, we require that there are d -edges from $\text{out}(h)$ to $\text{mid}(h)$ that establish an isomorphism between $\text{out}(h)$ and the graph obtained from $\text{mid}(h)$ by removing all α - and δ -edges. This means that $\text{out}(h) = \text{gr}(w)$. Third, it remains to require that $\text{in}(h)$ is isomorphic to g . To realize this, we require that $\text{in}(h) \in \mathcal{G}_\Gamma$ and that there are e -edges from $\text{in}(h)$ to $\text{mid}(h)$ that establish a bijection between $V_{\text{in}(h)}$ and the nodes of $\text{mid}(h)$ that have an incoming $\#$ -edge (thus representing a bijection between $V_{\text{in}(h)}$ and $V_g = \{a, a^2, \dots, a^n\}$). Since we wish this bijection to represent an isomorphism between $\text{in}(h)$ and g , we require for every $(x, \gamma, y) \in V_{\text{in}(h)} \times \Gamma \times V_{\text{in}(h)}$ that (x, γ, y) is an edge of $\text{in}(h)$ if and only if there exist nodes x', x'', y', y'' of $\text{mid}(h)$ such that

- (1) (x, e, x') and (y, e, y') are edges of h ,
- (2) (x'', δ, x') and (y'', δ, y') are edges of $\text{mid}(h)$,
- (3) x'' has an incoming $\$$ -edge in $\text{mid}(h)$, and
- (4) there is a directed path from x'' to y'' in $\text{mid}(h)$, of which the consecutive edge labels form a string in $a^*\gamma$.

This situation is sketched in Fig. 4. Condition (1) means that x and y correspond to substrings $\#a^i*$ and $\#a^j*$ of w (with $*$ in $\{\#, \$\}$), i.e., to nodes a^i and a^j of g , and conditions (2)-(4) mean that w has a substring $\$a^i\gamma a^j\$$, i.e., that (a^i, γ, a^j) is an edge of g . It should be clear that all these requirements can be expressed in MSO logic, and that the graph relation computed by H is enc_Γ . \square

⁵ We recall that the set of graphs $\text{gr}(w)$, where w is an arbitrary string over Ω , is MSO-definable, see for instance [3, Corollary 5.12] or [9, Example 2.1].

Lemma 2 is trivial from the point of view of Turing computability: if w encodes g , then both g and $gr(w)$ can be represented by w on the tape of a Turing machine. This is however based on the intuition that our encoding of graphs as strings is computable. Since the notion of MSO-computability discussed here uses graphs as datatype rather than strings, we were able to give a formal proof of that intuition. The reader may object that the formal proof is based on the intuition that the encoding of a string w as the graph $gr(w)$ is computable. One might then argue that the latter encoding is simpler than the former.

Traditionally, it has been shown that MSO logic is related to regularity, e.g., to regular string languages [2,6,12] and regular tree languages [4,13]. If one identifies regularity with computability by a finite-state machine, then this approach fails for MSO logic on graphs, because “no notion of finite graph automaton has been defined that would generalize conveniently finite automata on words and terms” ([3, Section 1.7]). For this reason, the MSO transducers of [3, Chapter 7] were proposed to play the role of finite-state transducers of graphs, and in the case of strings they indeed turned out to be equivalent to two-way finite-state transducers [5]. We have shown above how, dropping the finite-state condition, MSO logic is related to computability by any machine.

If, on the other hand, one identifies regularity with rationality, i.e., with a smallest class containing all finite sets of objects and closed under a number of natural operations on sets of objects (union, concatenation, and Kleene star in the case of string languages), then the class of all MSO-definable sets of graphs has a rational characterization [7]. Since the recursively enumerable string relations also have a rational characterization (as discussed in [8]), the question remains whether there is a natural rational characterization of the MSO-computable graph relations. Such a characterization would at least involve the operations of union, composition, and transitive closure of graph relations.

The above quote from [3, Section 1.7] refers to the non-existence of a finite-state graph automaton that accepts exactly the MSO-definable sets of graphs. In [11] a finite-state graph acceptor is introduced of which the computations are “tilings” of the input graphs (which have to be graphs of bounded degree). All “tiling-recognizable” sets of graphs accepted by these machines are MSO-definable, and the reverse is true for strings and trees. If we would allow the nodes of our graphs to have labels, then we could model the input graph $in(h)$ and the output graph $out(h)$ of a computation graph h by two special node labels rather than by v -edges. Then, similar to MSO-computability, we could define a graph relation to be “tiling-computable” by requiring the set H of computation graphs to be tiling-recognizable rather than MSO-definable. This leads to the following question for graphs of bounded degree: is every recursively enumerable graph relation tiling-computable? Note that, as shown in [11, Example 3.2(b)], the set of grids is tiling-recognizable.

Descriptive complexity theory investigates logics that characterize complexity classes. By Fagin’s theorem (see, e.g., [10, Theorem 5.1]), the complexity class NP equals

the set of problems that can be specified by existential second-order formulas. In terms of graphs, such a formula requires the existence of an extension of the input graph by additional labeled hyperedges (where a hyperedge is a sequence of nodes), such that the resulting (hyper)graph satisfies a first-order formula. In our notion of MSO-computability we require that the input graph is an induced subgraph of a graph that satisfies a monadic second-order formula, and we obtain all recursively enumerable problems.

We finally note that the notion of MSO-computability can easily be generalized to deal with arbitrary relational structures (cf. [3, Section 5.1]).

Declaration of competing interest

The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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That’s all folks! This was my last paper. Thank you, dear reader, and farewell.

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